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Fixed Point Theorems for Multifunctions in Topological Vector Spaces

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1. INTRODUCTION

In 1941 Kakutani [7] proved an important proposition that is a sufficient condition for the existence of fixed points for the multifunctions $F: S \to \mathfrak{p}(S)$, where S is a compact and convex subset of the space $X = \mathbb{R}^n$, satisfying the properties

(α) $F(x)$ is closed and convex, $\forall x \in S$;

($\alpha\alpha$) F is (u.s.c.)$_t$ on S.

This theorem is the starting point for many studies on the problem of the existence of fixed points in more general contexts and under weaker assumptions. Glicksberg [3] and Fan [2] improve Kakutani's theorem because for these authors X is a Hausdorff locally convex topological vector space; Himmelberg [5], in this more general context, succeeds in relaxing the compactness of S with the following condition:

($\alpha\alpha\alpha$) there exists a compact set C such that $F(S) \subset C$.

In [1] we obtain a fixed point theorem that improves the mentioned Himmelberg Theorem because the assumption ($\alpha\alpha$) is changed with the weaker condition

($\alpha\alpha$)* F has weakly closed graph.

Hadzic [4] and Idzik [6] study this problem without the condition that X is locally convex and they obtain two new theorems.

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Recently Tian [10] has stated some necessary and sufficient conditions for a multifunction F: $S \rightarrow \mathfrak{p}(S)$ to have a fixed point, where S is any subset of a Hausdorff locally convex topological vector space.

In this paper we prove two necessary and sufficient conditions for the existence of fixed points for a multifunction F: $S \rightarrow \mathfrak{p}(S)$ (cf. here Theorems II and III). In Theorem II we do not require that the Hausdorff topological vector space X is locally convex, while in the Theorem III we do.

Our Theorem II (cf. here Remark I) improves the mentioned propositions of Tian (cf. [10, Theorem 3]) and of Hadzic (cf. [4, Corollary 2]), while it extends Idzik's fixed point theorem [6], in the sense that there exist multifunctions that satisfy the conditions of our proposition but not the assumptions of Idzik's theorem.

Moreover our Theorem III improves Theorem 2 stated by us in [1].

Finally, in Section 4 we give a negative answer to the conjecture posed by Tian in [10], showing that it is not possible to use the techniques developed in [10] to extend the Lim fixed point theorem of [8] for nonexpansive and weakly inward multifunctions defined in an any subset of a uniformly convex Banach space.

2. Preliminaries

Let X be a Hausdorff topological vector space, S be a nonempty subset in X, and $\mathfrak{p}(X)$ be the family of all nonempty subsets of X. Given a multifunction F: $S \rightarrow \mathfrak{p}(X)$, we denote by $F(H)$, $F^{+}(K)$, and $F^{-}(K)$ the sets

$$F(H) = \bigcup_{x \in H} F(x),$$

$$F^{+}(K) = \{x \in S\colon F(x) \subset K\}$$

$$F^{-}(K) = \{x \in S\colon F(x) \cap K \neq \emptyset\},$$

where $H, K \subset X$.

A multifunction F: $S \rightarrow \mathfrak{p}(X)$ is said to be upper semicontinuous (in topological sense) (in short (u.s.c.)$_t$) on S if for every open subset A of X the set $F^{+}(A)$ is open in S, or, equivalently, if for every closed subset C of X the set $F^{-}(C)$ is closed in S.

Moreover, F is said to have closed graph if the set $\mathrm{Gr}(F) = \{(x, y) \in S \times X\colon y \in F(x)\}$ is closed in $X \times X$, or, equivalently, if for every net $(x_\delta)_\delta \subset S$, $x_\delta \rightarrow x$, $x \in S$, and for every net $(y_\delta)_\delta$, $y_\delta \in F(x_\delta)$, $y_\delta \rightarrow y$, $y \in S$ it follows $y \in F(x)$. On the other hand, as in [1], F is said to have weakly closed graph if for every net $(x_\delta)_\delta \subset S$, $x_\delta \rightarrow x$, $x \in S$, and for every net $(y_\delta)_\delta$, $y_\delta \in F(x_\delta)$, $y_\delta \rightarrow y$ it follows $\mathscr{L}(x, y) \cap F(x) \neq \emptyset$, where $\mathscr{L}(x, y) = \{x + \lambda(y - x)\colon \lambda \in [0, 1]\}$.

Given a point $x \in X$ and a convex (nonempty) set $D \subset X$, and denoting by $I_D(x)$ the set

$$I_D(x) = \{z \in X: \exists\, \lambda \in [0, 1[\ ,\ \lambda x + (1 - \lambda)z \in D\},$$

the multifunction $F: S \to \mathcal{P}(X)$ is said to be inward on $D \subset S$ if $F(x) \cap I_D(x) \neq \emptyset$, $\forall x \in D$, while F is said to be weakly inward on $D \subset S$ if $F(x) \cap \overline{I_D(x)} \neq \emptyset$, $\forall x \in D$ (where $\overline{I_D(x)}$ is the closure of the set $I_D(x)$).

In the case where X is a normed space, a multifunction $F: S \to \mathcal{P}(X)$ is said to be nonexpansive on S if $H(F(x), F(y)) \leq \|x - y\|$, $\forall x, y \in S$, where $H(F(x), F(y))$ is the Hausdorff distance between the sets $F(x)$ and $F(y)$.

The family of all neighbourhoods of zero in X being denoted by $\mathcal{U}(0)$, a set $H \subset X$ is called locally convex if for every $x \in H$ and for every $V \in \mathcal{U}(0)$ there exists a neighbourhood $U \in \mathcal{U}(0)$ such that $\mathrm{co}((x + U) \cap H) \subset x + V$ (where $\mathrm{co}((x + U) \cap H)$ is the convex hull of the set $(x + U) \cap H$); on the other hand, H is said to have Zima's property if for every $V \in \mathcal{U}(0)$ there exists a neighbourhood $U \in \mathcal{U}(0)$ such that $\mathrm{co}(U \cap (H - H)) \subset V$. Finally, the set H is said to be convexly totally bounded (in short (c.t.b.)) if for every $V \in \mathcal{U}(0)$ there exist a finite number of points $x_1, \ldots, x_n \in H$ and a finite number of convex subsets $C_1, \ldots, C_n$ of V such that

$$H \subset \bigcup_{i=1}^{n} (x_i + C_i).$$

First we state the following proposition:

THEOREM I. *Let X be a Hausdorff topological vector space and H be a nonempty subset of X.*
In these conditions we have that:

(p_1) *H has Zima's property $\Leftrightarrow$ $\overline{H}$ has Zima's property;*

(p_2) *H has Zima's property $\Rightarrow$ H is locally convex;*

(p_3) *H is locally convex and it has non empty interior $\Rightarrow$ X is a locally convex space.*

The sufficient part of the proposition (p_1) is trivial. In order to prove the necessary part, we fix a neighbourhood $V \in \mathcal{U}(0)$ and another neighbourhood $W \in \mathcal{U}(0)$ with the property

$$W + W \subset V. \tag{2.1}$$

Since H has Zima's property, there exists a neighbourhood $U \in \mathcal{U}(0)$ such that

$$\mathrm{co}(U \cap (H - H)) \subset W; \tag{2.2}$$

then, given $U' \in \mathcal{U}(0)$ with the property

$$U' + U' + U' \subset U, \tag{2.3}$$

we will prove the following inclusion:

$$\mathrm{co}(U' \cap (\overline{H} - \overline{H})) \subset \overline{\mathrm{co}(U \cap (H - H))}. \tag{2.4}$$

For this reason, if $y \in \mathrm{co}(U' \cap (\overline{H} - \overline{H}))$, we have

$$y = \sum_{i=1}^{n} \lambda_i z_i, \tag{2.5}$$

where

$$\sum_{i=1}^{n} \lambda_i = 1, \qquad \lambda_i \geq 0 \text{ and } z_i \in U' \cap (\overline{H} - \overline{H}), \qquad \forall i = 1, \ldots, n.$$

Now, given $I \in \mathcal{U}(0)$ we choose $J \in \mathcal{U}(0)$ such that

$$J + J \subset I, \tag{2.6}$$

and $J_i \in \mathcal{U}(0)$, $i = 1, \ldots, n$, with the property

$$J_1 + \ldots + J_n \subset J \cap U'. \tag{2.7}$$

For every $i \in \{1, \ldots, n\}$, since $z_i = k_i - k_i^*$, $k_i, k_i^* \in \overline{H}$, there exist $h_i, h_i^* \in H$ and $j_i, j_i^* \in J_i$ in the following way:

$$h_i = k_i + j_i, \qquad h_i^* = k_i^* + j_i^*. \tag{2.8}$$

By (2.5), and taking (2.8) into account, it follows that

$$y + r = \sum_{i=1}^{n} \lambda_i (h_i - h_i^*), \tag{2.9}$$

where $r = \sum_{i=1}^{n} \lambda_i (j_i - j_i^*) \in I$ (cf. here (2.6) and (2.7)). On the other hand, since $z_i \in U'$, $i = 1, \ldots, n$, by (2.8) we have that (cf. here (2.7) and (2.3)) $h_i - h_i^* \in J_i + J_i + U' \subset U$, $\forall i = 1, \ldots, n$, and then $\sum_{i=1}^{n} \lambda_i (h_i - h_i^*) \in \mathrm{co}(U \cap (H - H))$. Therefore, taking (2.9) into account, (2.4) is proved.

Moreover, by using

$$\overline{\mathrm{co}(U \cap (H - H))} \subset \mathrm{co}(U \cap (H - H)) + W,$$

and by (2.2), (2.1), and (2.4), it follows that $\mathrm{co}(U' \cap (\overline{H} - \overline{H})) \subset V$.

In order to prove the proposition (p_2) we fix a point $x \in H$ and a neighbourhood $V \in \mathcal{U}(0)$. Since H has Zima's property, it is possible to find a neighbourhood $U \in \mathcal{U}(0)$ such that

$$\mathrm{co}(U \cap (H - x)) \subset V. \tag{2.10}$$

If $z \in \mathrm{co}((x + U) \cap H)$, it follows that $z = \sum_{i=1}^{m} \alpha_i q_i$, where

$$\sum_{i=1}^{m} \alpha_i = 1, \qquad \alpha_i \geq 0 \text{ and } q_i \in (x + U) \cap H, \qquad \forall i = 1, \ldots, m,$$

and then (cf. (2.10))

$$z - x = \sum_{i=1}^{m} \alpha_i (q_i - x) \in \mathrm{co}(U \cap (H - x)) \subset V;$$

thus

$$\mathrm{co}((x + U) \cap H) \subset x + V.$$

Finally, we will prove the proposition (p_3). Put $W \in \mathcal{U}(0)$ and $x \in \mathring{H}$ ($\mathring{H}$ denotes the interior of H); since H is locally convex, there exists a neighbourhood $U \in \mathcal{U}(0)$, $x + U \subset H$, such that $\mathrm{co}(x + U) \subset x + W$ and so the open set $\mathrm{co}\, U$ is included in W. Therefore, X has a base of convex neighbourhoods of zero.

3. The Existence of Fixed Points for Multifunctions

In this section we prove two necessary and sufficient conditions for the existence of fixed points for multifunctions.

Theorem II. *Let X be a Hausdorff topological vector space, S be a non empty subset of X and F: $S \to \mathfrak{p}(X)$ be a multifunction with the properties*

(i) *$F(x)$ is closed and convex, $\forall x \in S$;*

(ii) *F has closed graph;*

(iii) *the set $\overline{F(S)}$ is locally convex.*

Under these conditions, the multifunction F has a fixed point if and only if there exists a compact and convex subset K of S such that

$$F(x) \cap K \neq \emptyset, \qquad \forall x \in K. \tag{3.1}$$

The necessary part is trivial because if $x^* \in S$ is such that $x^* \in F(x^*)$, putting $K = \{x^*\}$, (3.1) is satisfied.

In order to prove the sufficient part, we define a multifunction $G: K \to \mathfrak{p}(K)$, by putting

$$G(x) = F(x) \cap K, \qquad \forall x \in K. \tag{3.2}$$

We note that, by (i) and by the compactness and convexity of the set K, the multifunction G has compact and convex values. Moreover, G is upper semicontinuous (in topological sense) on S; in fact, since G is a multifunction that assumes values in a Hausdorff topological compact space, it is sufficient to prove that G has closed graph. For this reason we fix a net $(x_\delta)_\delta \subset K$ that converges to a point x_θ and a net $(y_\delta)_\delta$, $y_\delta \in G(x_\delta)$, $y_\delta \to y_\theta$; from $y_\delta \in F(x_\delta)$, $\forall \delta$, and by (ii) it follows that $y_\theta \in F(x_\theta)$ and hence $y_\theta \in G(x_\theta)$.

Now we observe that the set $\overline{G(K)}$ is compact, and, as we will prove, it is locally convex. In fact, given $z \in \overline{G(K)}$ and a neighbourhood $V \in \mathcal{U}(0)$, since $\overline{G(K)} \subset \overline{F(S)}$ and $\overline{F(S)}$ is locally convex, there exists a neighbourhood $U \in \mathcal{U}(0)$ such that $\mathrm{co}((z + U) \cap \overline{F(S)}) \subset z + V$, from which it follows that $\mathrm{co}((z + U) \cap \overline{G(K)}) \subset z + V$.

Finally, taking Proposition 4.5 of [6] into account, we can say that the set $\overline{G(K)}$ is c.t.b. Therefore the multifunction G satisfies the conditions of Theorem 4.3 of [6]: thus there exists a point $x^* \in K$ such that $x^* \in G(x^*)$, and so (cf. (3.2)) $x^* \in F(x^*)$.

Remark I. We wish to note that our Theorem II strictly contains Theorem 3 stated by Tian in [10]. In fact, it is easy to see that the assumptions of Tian's Theorem imply the conditions of our proposition; on the other hand, there exist multifunctions that satisfy the hypothesis of our theorem but do not satisfy the assumptions of the above-mentioned Tian's Theorem even if X is a Hausdorff locally convex topological vector space. This is evident by taking the following example into account.

EXAMPLE 1. Let $X = \mathbb{R}^2$, $S = \{(t, 0): t \in \mathbb{R}\}$, and $F: S \to \mathfrak{p}(X)$ be the multifunction defined putting

$$F(t, 0) = \{(x, y) \in \mathbb{R}^2: y = tx\}, \qquad \forall (t, 0) \in S. \tag{3.3}$$

It is simple to observe that the multifunction F has closed and convex values, and that the set $\overline{F(S)}$ is locally convex. In order to prove that F has closed graph we note that if $\{(t_n, 0)\}_n \subset S$ is a sequence that converges to $(t^*, 0)$ and $\{(x_n, y_n)\}_n$ is a sequence that converges to a point $(x^*, y^*) \in \mathbb{R}^2$ such that $(x_n, y_n) \in F(t_n, 0)$, $\forall n \in \mathcal{N}$, since $y^* = \lim_{n \to +\infty} t_n x_n = t^* x^*$, we have (cf. here (3.3)) $(x^*, y^*) \in F(t^*, 0)$.

On the other hand, the multifunction F is not (u.s.c.)$_t$ on S, as Tian assumes in his proposition. In fact, if we choose the open set $A = \{(x, y) \in \mathbb{R}^2 : y \in]-1, 1[\}$, it is trivial to see that $F^+(A) = \{(0, 0)\}$ is not open in S.

Moreover, Theorem II strictly contains also Corollary 2 proved by Hadzic in [4]. First, taking the necessary part of the proposition (p_1) and the proposition (p_2) of Theorem I into account, we can say that every multifunction satisfying the conditions of Corollary 2 verifies the assumptions of our Theorem II, but not vice versa, because for us the set S is not necessarily closed and convex.

Finally, our Theorem II extends the above-mentioned Idzik's fixed point theorem [6], in the sense that there exist multifunctions satisfying the hypothesis of our theorem but not satisfying the assumptions required by Idzik: for this it is sufficient to observe that for Idzik the multifunctions are defined on a closed and convex set (cf. [6, Corollary 2.6]).

THEOREM III. *Let X be a Hausdorff locally convex topological vector space, S be a non empty subset of X, and $F: S \to p(X)$ be a multifunction with the properties*

(i) *$F(x)$ is closed and convex, $\forall x \in S$;*

(α) *F has weakly closed graph.*

Under these conditions, the multifunction F has a fixed point if and only if there exists a compact and convex subset K of S satisfying (3.1).

The necessary part is trivial. In order to prove the sufficient part we define the multifunction $G: K \to p(K)$ as in (3.2) and we will prove that it has weakly closed graph. Given a net $(x_\delta)_\delta \subset K$, $x_\delta \to x_\theta$, and another net $(y_\delta)_\delta$, $y_\delta \in G(x_\delta)$, $\forall \delta$, $y_\delta \to y_\theta$, then by the hypothesis (α) we have that $\mathcal{L}(x_\theta, y_\theta) \cap F(x_\theta) \neq \emptyset$; since K is a closed and convex set, we obtain that $\mathcal{L}(x_\theta, y_\theta) \cap G(x_\theta) \neq \emptyset$.

Moreover, the multifunction G has closed and convex values and the set $G(K)$ is included in the compact set K, therefore G satisfies the assumptions of Theorem II of [1]. Then F has a fixed point.

Remark II. We wish to point out that there exist multifunctions that satisfy the hypothesis of Theorem III, but do not satisfy all the conditions of Theorem II. This is easy to observe, taking the multifunction

$F\colon S \to \mathfrak{p}(S)$, $S = [0, 2]$, defined putting

$$F(x) = \begin{cases} \{1\}, & x \in [0, 2[\\ \{2\}, & x = 2, \end{cases}$$

into account.

Moreover, we wish to note that Theorem III improves our Theorem II of [1].

4. Answer to a Conjecture Posed by G. Tian

Finally, we obtain the following.

THEOREM IV. *Let X be a uniformly convex Banach space, S be a non empty subset of X, and $F\colon S \to \mathfrak{p}(X)$ be a multifunction with the properties*

(β) *$F(x)$ is compact, $\forall x \in S$;*

($\beta\beta$) *F is non expansive on S;*

($\beta\beta\beta$) *there exists a closed, convex, and bounded subset C of S such that $F(x) \subset \overline{I_C(x)}$, $\forall x \in C$.*

In these conditions, F has a fixed point.

The proof follows immediately from Lim's fixed point Theorem (cf. [9, Theorem 8]).

Remark III. We wish to observe that it is not possible to change our assumption ($\beta\beta\beta$) with the weaker hypothesis:

($\beta\beta\beta$)* there exists a closed, convex, and bounded subset C of S in which F is inward.

In fact, even in the case $X = \mathbb{R}$, there exist multifunctions F, defined in a subset S of $\mathbb{R}$ satisfying the conditions (β), ($\beta\beta$), and ($\beta\beta\beta$)* but not having a fixed point, as the following example shows.

EXAMPLE 2. Let $X = \mathbb{R}$, $S =]1, 2[$, and $F\colon S \to \mathfrak{p}(X)$ be the multifunction defined putting

$$F(x) = [0, 1] \cup [2, 3], \qquad \forall x \in S.$$

The multifunction F has compact values, it is nonexpansive, and, moreover, put $C = [\frac{5}{4}, \frac{7}{4}]$; we have $F(x) \cap I_C(x) \neq \varnothing$, $\forall x \in C$, and therefore F also satisfies the property ($\beta\beta\beta$)*.

Taking the above remark into account, we can give a negative answer to the conjecture posed by Tian in [10].

In fact, in Tian's opinion it was possible, using the techniques developed in his note, to extend fixed point theorems, such as Lim's fixed point theorem (cf. [8, Theorem 1]), for nonexpansive and weakly inward multifunctions defined in a nonclosed and nonconvex domain.

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